Baseball simulation: finding the collision point between sampled frames

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1 Objectives

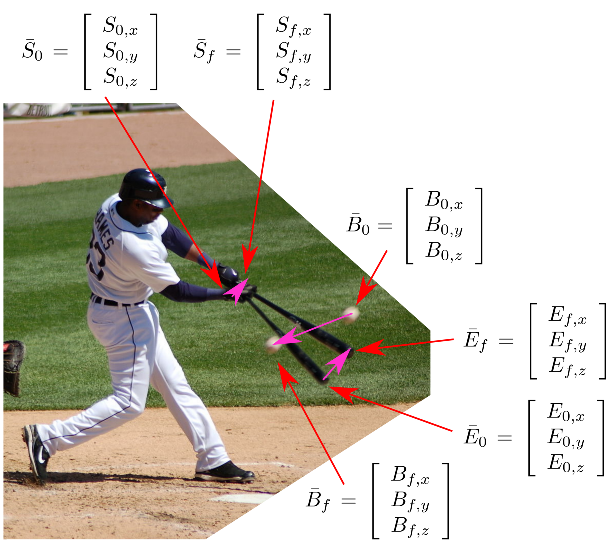


Figure 1: Baseball hit showing the position of the ball and bat in two sampled frames before computing the collision time and location

Begin with 6 vectors, $\bar{B}_0$ and $\bar{B}_f$ the initial and final positions of the baseball, $\bar{E}_0$ and $\bar{E}_f$ the initial and final positions of the end of the bat, and $\bar{S}_0$ and $\bar{S}_f$ the initial and final positions of the knob of the bat. Given linear interpolation between these vectors, we have two objectives:

1. Find the distance between the ball and bat as an explicit function of time and known position vectors, $d(t) = f(t, \bar{B}_0, \bar{B}_f, \bar{E}_0, \bar{E}_f, \bar{S}_0, \bar{S}_f)$
2. Find the time of impact (if any) between t_0 and $t_0 + \Delta t$, i.e. find t such that $d(t) = 0$.

Expressions for objectives 1 and 2 are at the ends of sections 2 and 5 respectively. See Eq (6) (objective **??**) and Eqs (15f) or (16f) (objective 2).

2 Problem setup $d(t)$

Define $\bar{B}$, $\bar{S}$, and $\bar{E}$ as functions of t linearly interpolating between the initial and final values. This produces t dependence of the form $\bar{B} = (\bar{B_f} - \bar{B}_0)\frac{t}{\Delta t} + \bar{B}_0$. More explicitly:

$$\bar{B} = \left(\begin{bmatrix} B_{f,x} \\ B_{f,y} \\ B_{f,z} \end{bmatrix} - \begin{bmatrix} B_{0,x} \\ B_{0,y} \\ B_{0,z} \end{bmatrix} \right) \frac{t}{\Delta t} + \begin{bmatrix} B_{0,x} \\ B_{0,y} \\ B_{0,z} \end{bmatrix} \tag{1}$$

From Wolfram Alpha described in detail in section 7 or the geometrical derivation in section 6, the distance between a point and a line (minus radii of the ball and the bat) is given by

$$d(t) = \frac{\left|(\bar{B} - \bar{E}) \times (\bar{B} - \bar{S})\right|}{\left|\bar{S} - \bar{E}\right|} - r_{bat} - r_{ball} \tag{2}$$

where the single bars refer to the 2nd norm (or Euclidean norm) of the argument vector $\bar{\lambda}$ with elements 1 to n, or in this case 1 to 3 (for 3 dimensional baseball).

$$\left|\bar{\lambda}\right| = \sqrt{\bar{\lambda}^T \bar{\lambda}} = \sqrt{\sum_i \lambda_i^2} = \sqrt{\lambda_0^2 + \cdots + \lambda_n^2} \tag{3}$$

Now rewrite $\bar{B} - \bar{E}$ as a function of t:

$$\begin{aligned} \bar{B} - \bar{E} &= (\bar{B_f} - \bar{B}_0)\frac{t}{\Delta t} + \bar{B}_0 - (\bar{E_f} - \bar{E}_0)\frac{t}{\Delta t} - \bar{E}_0 \\ &= (\bar{B_f} - \bar{B}_0 - \bar{E_f} + \bar{E}_0)\frac{t}{\Delta t} + \bar{B}_0 - \bar{E}_0 \\ &= \bar{a}t + \bar{b} \end{aligned} \tag{4}$$

Following that same sequence of rearranging and group the radii in one variable:

$$\begin{aligned} \bar{B} - \bar{S} &= (\bar{B_f} - \bar{B}_0 - \bar{S_f} + \bar{S}_0)\frac{t}{\Delta t} + \bar{B}_0 - \bar{S}_0 \\ &= \bar{c}t + \bar{d} \end{aligned} \tag{5a}$$

$$\begin{aligned} \bar{S} - \bar{E} &= (\bar{S_f} - \bar{S}_0 - \bar{E_f} + \bar{E}_0)\frac{t}{\Delta t} + \bar{S}_0 - \bar{E}_0 \\ &= \bar{e}t + \bar{f} \end{aligned} \tag{5b}$$

$$r = r_{ball} + r_{bat} \tag{5c}$$

Now we have d as an explicit function of t and collected variables:

$$d(t) = \frac{\left|(\bar{a}t + \bar{b}) \times (\bar{c}t + \bar{d})\right|}{\left|\bar{e}t + \bar{f}\right|} - r \tag{6}$$

3 Simplify the denominator to the root of a polynomial

Begin with the denominator because it is less complex. The denominator is the length of a vector as a function of t and can be expressed as the square root of a 2nd order polynomial:

$$\begin{aligned} \left|\bar{S}-\bar{E}\right| &= \left|\bar{e}t+\bar{f}\right| = \sqrt{\sum(e_i t + f_i)^2} \\ &= \sqrt{\sum(e_i^2 t^2 + 2e_i f_i t + f_i^2)} \\ &= \sqrt{\begin{bmatrix} \sum e_i^2 & 2\sum e_i f_i & \sum f_i^2 \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}} \end{aligned} \tag{7}$$

Define $\bar{\alpha}$ and t for convenience:

$$\begin{aligned} \bar{\alpha} &= \sum \begin{bmatrix} e_i^2 & 2e_i f_i & f_i^2 \end{bmatrix} \\ \bar{t} &= \begin{bmatrix} t^2 \\ t^1 \\ t^0 \end{bmatrix} = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} \end{aligned} \tag{8}$$

$$\left|\bar{S}-\bar{E}\right| = \sqrt{\bar{\alpha}\bar{t}} \tag{9}$$

Remember coefficients to polynomials ($\bar{\alpha}$ here, and $\bar{\beta}$, defined later in Section 4) are defined as *horizontal* arrays or *row vectors*. $\bar{t}$ is a *vertical* array or *column vector* of decreasing powers of t^{n-i} for $i=[0,n]$ where n is the length of the coefficient array minus 1.

4 Simplify the numerator to the root of a polynomial

This will be more complicated but it's just a lot of algebraic rearrangement:

$$\begin{aligned} \left|\left(\bar{a}t+\bar{b}\right)\times\left(\bar{c}t+\bar{d}\right)\right| &= \left|(\bar{a}\times\bar{c})\,t^2 + \{(\bar{b}\times\bar{c}) + (\bar{a}\times\bar{d})\}t + (\bar{b}\times\bar{d})\right| \\ &= \left\| \begin{bmatrix} \vdots & \vdots & \vdots \\ (\bar{a}\times\bar{c}) & (\bar{b}\times\bar{c}) + (\bar{a}\times\bar{d}) & (\bar{b}\times\bar{d}) \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix} \right\| \\ &= \left|\bar{\bar{X}}\bar{t}\right| \end{aligned} \tag{10}$$

Now here various cross products of $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$ are used to define the columns of $\bar{\bar{X}}$. Remember that $\bar{\bar{X}}\bar{t}$ is a column vector of 3 elements of the form $\left(x_{i,1}t^2 + x_{i,2}t + x_{i,3}\right)$:

$$
\begin{aligned}
\left|\left(\bar{a}t+\bar{b}\right)\times\left(\bar{c}t+\bar{d}\right)\right| &= \sqrt{\left(\bar{\bar{X}}\bar{t}\right)^T\left(\bar{\bar{X}}\bar{t}\right)} \\
&= \sqrt{\sum_i^{1,2,3}\left(x_{i,1}t^2+x_{i,2}t+x_{i,3}\right)^2} \\
&= \sqrt{\sum_i^{1,2,3} x_{i,1}^2t^4+2x_{i,1}x_{i,2}t^3+\left(x_{i,2}^2+2x_{i,1}x_{i,3}\right)t^2+2x_{i,2}x_{i,3}t+x_{i,3}^2} \qquad (11)\\
&= \sqrt{\sum_n^{[0,4]}\beta_{4-n}t^n} \\
&= \sqrt{\bar{\beta}\bar{t}}
\end{aligned}
$$

With $\bar{\beta}$ and $\bar{t}$ as:

$$
\begin{aligned}
\bar{\beta} &= \sum_i^{1,2,3}\begin{bmatrix} x_{i,1}^2 & 2x_{i,1}x_{i,2} & \left(x_{i,2}^2+2x_{i,1}x_{i,3}\right) & 2x_{i,2}x_{i,3} & x_{i,3}^2 \end{bmatrix} \\
\bar{t} &= \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t^1 \\ t^0 \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}
\end{aligned} \qquad (12)
$$

Notice that i only appears in the first subscript of x (i.e. $x_{i,1}$ but never as $x_{1,i}$) so the summation for $\bar{\beta}$, Eq (12), is just using i to move down the columns of $\bar{\bar{X}}$ one row at a time.

5 Set $d(t) = 0$ and solve for t

Before we begin we will reindex $\bar{\alpha}$ (which has 3 elements) so it has a length of 5 and the first two elements are 0:

$$\bar{\alpha} = \begin{bmatrix} 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix} \qquad (13)$$

And for the rest of this document $\bar{t}$ has a length of 5. Notice that with this new $\bar{\alpha}$ and a $\bar{t}$ of length 5, $\bar{\alpha}\bar{t}$ produces the same result as it did with arrays of length 3, just with two additional terms of 0 in the summation $\left(0t^4+0t^3+\ldots\right)$

$$\bar{t} = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \qquad (14)$$

Now we can solve this very concisely with vectors to come up with a 4th order polynomial = 0:

$$d = 0 = \frac{\left|(\bar{a}t + \bar{b}) \times (\bar{c}t + \bar{d})\right|}{\left|\bar{e}t + \bar{f}\right|} - r = \frac{\sqrt{\bar{\beta}\bar{t}}}{\sqrt{\bar{\alpha}\bar{t}}} - r \tag{15a}$$

$$r^2 = \frac{\bar{\beta}\bar{t}}{\bar{\alpha}\bar{t}} \tag{15b}$$

$$r^2 \left(\bar{\alpha}\bar{t}\right) = \bar{\beta}\bar{t} \tag{15c}$$

$$0 = \bar{\beta}\bar{t} - r^2 \left(\bar{\alpha}\bar{t}\right) \tag{15d}$$

$$0 = \bar{\beta}\bar{t} - \left(r^2\bar{\alpha}\right)\bar{t} \tag{15e}$$

$$0 = \left(\bar{\beta} - r^2\bar{\alpha}\right)\bar{t} \tag{15f}$$

Or we can use summations to obtain the same result much less elegantly:

$$d = 0 = \frac{\left|(\bar{a}t + \bar{b}) \times (\bar{c}t + \bar{d})\right|}{\left|\bar{e}t + \bar{f}\right|} - r = \frac{\sqrt{\sum_n^{[0,4]} \beta_{4-n} t^n}}{\sqrt{\sum_n^{[0,4]} \alpha_{4-n} t^n}} - r \tag{16a}$$

$$r^2 = \frac{\sum_n^{[0,4]} \beta_{4-n} t^n}{\sum_n^{[0,4]} \alpha_{4-n} t^n} \tag{16b}$$

$$r^2 \sum_n^{[0,4]} \alpha_{4-n} t^n = \sum_n^{[0,4]} \beta_{4-n} t^n \tag{16c}$$

$$0 = \sum_n^{[0,4]} \beta_{4-n} t^n - r^2 \sum_n^{[0,4]} \alpha_{4-n} t^n \tag{16d}$$

$$0 = \sum_n^{[0,4]} \beta_{4-n} t^n - r^2 \alpha_{4-n} t^n \tag{16e}$$

$$0 = \sum_n^{[0,4]} \left(\beta_{4-n} - r^2 \alpha_{4-n}\right) t^n \tag{16f}$$

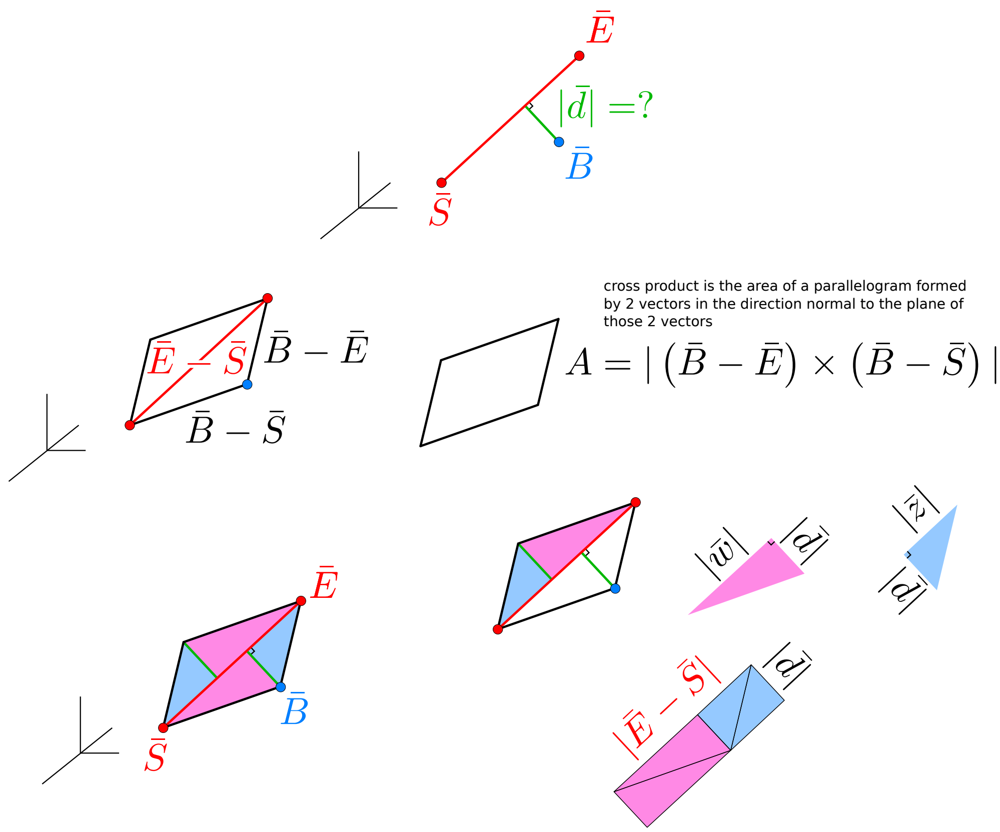
Please notice that the only difference here is the notation, the meaning is identical, e.g. eqs. (15a) to (15f) are exactly identical to eqs. (16a) to (16f) respectively.

The final result in equation (15f) or (16f) is a 4th order polynomial = 0. This can be solved with any number of previously developed, efficient polynomial root finders such as NumPy (numpy.roots) using the coefficients $\left(\bar{\beta} - r^2\bar{\alpha}\right)$. The coefficient arrays, α and β, have been defined so that increasing index corresponds to a decreasing power of t in keeping with the convention required by built in root finders in NumPy and MatLab. Of the real values of t in the range $[0, \Delta t]$, we are looking for minimum, i.e. the first "kiss."

The problem is now solved; the following sections explain the origin of Eq (2).

6 Geometrical derivation of $|\bar{d}|$

The goal in sections 6 and 7 is to derive Eq (2) so as to understand it's origin and verify it's accuracy. Section 6 employs mostly geometrical arguments while section 7 is entirely algebraic. The algebraic derivation was copied from Wolfram Alpha but the geometric derivation is my own. When I first solved the bat problem, I copied the Wolfram Alpha algebraic solution.



Remember the d of interest is the length of the vector $\bar{d}$, $d = |\bar{d}|$. Notice that $|\bar{w} + \bar{z}| = |\bar{w}| + |\bar{z}|$ since $\bar{w}$ and $\bar{z}$ are parallel $\|$:

$$A = |\bar{w}|\,|\bar{d}| + |\bar{z}|\,|\bar{d}| \tag{17a}$$
$$= |\bar{w} + \bar{z}|\,|\bar{d}| \tag{17b}$$
$$= |\bar{S} - \bar{E}|\,|\bar{d}| \tag{17c}$$

Now in Eq (17a) we set the area, A, to the sum of the areas of the two pink triangles ($|\bar{w}|\,|\bar{d}|$) and the two blue triangles ($|\bar{z}|\,|\bar{d}|$) and the cross product expression for A given in the figure:

$$A = |\bar{S} - \bar{E}|\,|\bar{d}| = \left|(\bar{B} - \bar{E}) \times (\bar{B} - \bar{E})\right| \tag{18}$$

Take the two expressions for A and solve for $|\bar{d}|$:

$$d = \left|\bar{d}\right| = \frac{\left|\left(\bar{B} - \bar{E}\right) \times \left(\bar{B} - \bar{E}\right)\right|}{\left|\bar{S} - \bar{E}\right|} \tag{19}$$

7 Algebraic derivation of $|\bar{d}|$

What follows is taken almost entirely from Wolfram Alpha, hopefully with notation improved for clarity and consistency with this document. Beginning with $\bar{E}$, $\bar{S}$, and $\bar{B}$, we would like to find the shortest distance between $\bar{B}$ and a line connecting $\bar{E}$ and $\bar{S}$. This is the simple static problem (no functions of t), the solution of which appears in a slightly modified form as Eq (2).

The line connecting $\bar{E}$ and $\bar{S}$ may be defined by a parametric equation and a dummy variable λ:

$$\begin{gathered}\bar{L} = \left(\bar{E} - \bar{S}\right)\lambda + \bar{S} \\ \bar{L} = \left(\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} - \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}\right)\lambda + \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}\end{gathered} \tag{20}$$

$$\left|\bar{d}(\lambda)\right| = \left|\bar{L}(\lambda) - \bar{B}\right| \tag{21}$$

Find the minimum of $\left|\bar{d}(\lambda)\right|$ by setting $\frac{\partial|\bar{d}|^2}{\partial\lambda} = 0$ and solving for λ:

$$\left|\bar{d}(\lambda)\right| = \left|\left(\bar{E} - \bar{S}\right)\lambda + \bar{S} - \bar{B}\right| \tag{22}$$

$$\left|\bar{d}(\lambda)\right|^2 = \sum_i \left[(E_i - S_i)\lambda + (S_i - B_i)\right]^2 \tag{23}$$

$$\frac{\partial\left|\bar{d}\right|^2}{\partial\lambda} = \sum_i \left[(E_i - S_i)\lambda + (S_i - B_i)\right](E_i - S_i) = 0 \tag{24}$$

$$\left|\left(\bar{E} - \bar{S}\right)\right|^2\lambda + \left(\bar{S} - \bar{B}\right) \bullet \left(\bar{E} - \bar{S}\right) = 0 \tag{25}$$

Solve for λ:

$$\lambda = -\frac{\left(\bar{S} - \bar{B}\right) \bullet \left(\bar{E} - \bar{S}\right)}{\left|\left(\bar{E} - \bar{S}\right)\right|^2} \tag{26}$$

Plug in λ and solve for $\left|\bar{d}\right|^2$:

$$
\begin{aligned}
\left|\bar{d}\right|^2 &= \left|\left(\bar{E}-\bar{S}\right)\lambda+\bar{S}-\bar{B}\right|^2 \\
&= \left(\bar{S}-\bar{B}\right)\bullet\left(\bar{S}-\bar{B}\right) \\
&\quad -2\left(\bar{E}-\bar{S}\right)\bullet\left(\bar{S}-\bar{B}\right)\tfrac{(\bar{E}-\bar{S})\bullet(\bar{S}-\bar{B})}{(\bar{E}-\bar{S})\bullet(\bar{E}-\bar{S})} \\
&\quad +\left(\bar{E}-\bar{S}\right)\bullet\left(\bar{E}-\bar{S}\right)\left[\tfrac{(\bar{E}-\bar{S})\bullet(\bar{S}-\bar{B})}{(\bar{E}-\bar{S})\bullet(\bar{E}-\bar{S})}\right]^2 \\
&= \left|\bar{S}-\bar{B}\right|^2-2\tfrac{\left[(\bar{E}-\bar{S})\bullet(\bar{S}-\bar{B})\right]^2}{\left|\bar{E}-\bar{S}\right|^2}+\left|\bar{E}-\bar{S}\right|^2\tfrac{\left[(\bar{E}-\bar{S})\bullet(\bar{S}-\bar{B})\right]^2}{\left|\bar{E}-\bar{S}\right|^4} \\
&= \tfrac{\left|\bar{S}-\bar{B}\right|^2\left|\bar{E}-\bar{S}\right|^2-\left[(\bar{E}-\bar{S})\bullet(\bar{S}-\bar{B})\right]^2}{\left|\bar{E}-\bar{S}\right|^2}
\end{aligned}
\tag{27}
$$

Using the vector quadruple product (28):

$$
\left|\bar{x}\times\bar{y}\right|^2=\left|\bar{x}\right|^2\left|\bar{y}\right|^2-\left(\bar{x}\bullet\bar{y}\right)^2 \tag{28}
$$

To simplify the numerator (remember $\left|\bar{S}-\bar{E}\right|=\left|\bar{E}-\bar{S}\right|$):

$$
\begin{aligned}
\left|\bar{d}\right|^2 &= \frac{\left|\left(\bar{B}-\bar{E}\right)\times\left(\bar{B}-\bar{S}\right)\right|^2}{\left|\bar{E}-\bar{S}\right|^2} \\
\left|\bar{d}\right| &= \frac{\left|\left(\bar{B}-\bar{E}\right)\times\left(\bar{B}-\bar{S}\right)\right|}{\left|\bar{S}-\bar{E}\right|}
\end{aligned}
\tag{29}
$$